Casimir Force between Vortex Matter in Anisotropic and Layered Superconductors

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We present a new approach to calculate the attractive long range vortex-vortex interaction of the van der Waals type present in anisotropic and layered superconductors. The mapping of the statistical mechanics of vortex lines onto the imaginary time quantum mechanics of two dimensional charged bosons allows us to define a 2D Casimir problem: Two half-spaces of (dilute) vortex matter separated by a gap of width R are mapped to two dielectric half-planes of charged bosons interacting via a massive gauge field. We determine the attractive Casimir force between the two half-planes and show, that it agrees with the pairwise summation of the van der Waals force between vortices previously found by Blatter and Geshkenbein [Phys. Rev. Lett. **77**, 4958 (1996)].

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I. INTRODUCTION

Within the Shubnikov phase of type II superconductors the applied magnetic field enters the sample in the form of flux lines. The standard mean-field type calculation[1] shows that in an isotropic material two straight vortices repel at all distance scales, with an interaction strength $V_{\rm rep} = 2\varepsilon_0 K_0(R/\lambda)$, $\varepsilon_0 = (\Phi_0/4\pi\lambda)^2$ being the basic energy scale in the vortex matter, K_0 is the zero-order modified Bessel function, R is the inter-vortex distance, and λ the magnetic penetration length ($\Phi_0 = hc/2e$ denotes the flux quantum). However, it has recently been shown[2] that in layered and strongly anisotropic superconductors the thermal fluctuations of the flux lines give rise to a long range attraction $V_{\rm vdW} \sim -(T/d)(\lambda/R)^4$ of the van der Waals type between the vortices, where T denotes the temperature and d is the interlayer separation. The strongly fluctuating and layered high temperature superconductors are particularly well suited to exhibit this attractive component in the vortex-vortex interaction. Alternatively, the attraction is induced through static vortex distortions due to an underlying pinning landscape, an effect recently studied by Mukherji and Nattermann[3] and by Volmer *et al.*[4]

Following a suggestion of Nelson[5], the statistical mechanics of vortices can be mapped to the imaginary time quantum mechanics of two-dimensional (2D) bosons. The particular interaction between the flux lines renders the bosons charged (with a charge screened on the scale of the London penetration depth λ). This type of long range interaction can be formulated in terms of a massive gauge field theory[6–8]. Within the resulting 2D massive electrodynamics, the vortex matter acts as a dielectric medium and we can define a Casimir problem, see Fig. 1: Under the vortex $\to$ boson mapping two half-spaces of vortices separated by a gap of width R act as two dielectric planar media which attract each other via a Casimir force.

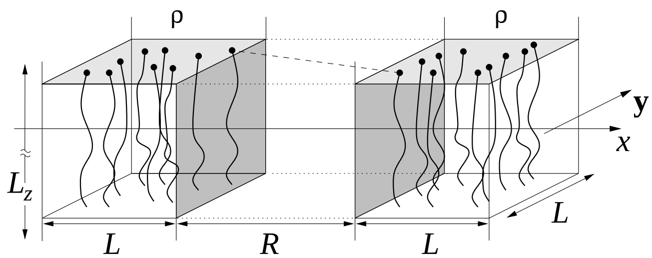

FIG. 1. Geometry used in the calculation of the Casimir force between two half spaces of vortex matter separated by a gap of width R. Shown are two parallel cubes of size $L_z \times L^2$ at a distance R with vortex density ρ ($L_z \to \infty$). We choose a coordinate system with the z-axis parallel to the vortex lines and the x-axis perpendicular to the planes of the gap (dark shaded planes). The statistical mechanics of the flux lines is mapped to the imaginary time quantum mechanics of two dimensional (2D) bosons (solid circles). The cut through the vortex lines (lightly shaded planes) defines a time slice of the 2D bosons. The Casimir energy produced by the dielectric action of the two media is calculated by summing over the ground state energies of the fluctuating massive electromagnetic field mediating the interaction between the 2D bosons. Using the technique of pairwise summation (dashed line) valid for rarefied media (i.e., $\rho \to 0$), we can relate the macroscopic Casimir effect with the microscopic van der Waals attraction between the particles involved. Picture not to scale.

In the present work, we determine the Casimir force between two dielectric half planes of charged bosons. For dilute media, the macroscopic Casimir force can be related to the microscopic van der Waals force between the media's constituents via pairwise summation. Here, we present a derivation of the van der Waals force between vortex lines via this alternative route, calculating first the Casimir force between two dielectric planar media in the boson picture and then reconstructing the van der Waals force in the reductionist way.

Since the discovery of the Casimir effect[9] in 1948, sev-

eral hundred papers have dealt with this phenomenon, disseminating its fascination into many branches of physics[10–17]. Casimir forces between macroscopic bodies are a quantum effect caused by a shift in the zero-point energy of gauge field fluctuations such as the electromagnetic one. The Casimir effect is bound to a number of system properties, such as topology and dielectric permittivity, and the reduction to an analogous van der Waals interaction is not always possible. However, the interpretation in terms of a van der Waals attraction is possible for the case of rarefied media and appropriate geometries, such as the parallel plate setup[17]. In our derivation of the van der Waals force from the Casimir effect we will make use of these special conditions.

In the following, we discuss the relationship between the Casimir effect and the van der Waals interaction within a path integral formulation (Sec. II) before deriving the appropriate action for the 2D charged bosons from the London functional describing the vortices (Sec. III). In section IV, we briefly review the derivation of the van der Waals force in the original vortex language and then proceed with the calculation of the 2D Casimir effect in section V, the main section of the paper containing the new results.

II. CASIMIR VERSUS VAN DER WAALS

We consider two parallel material slabs made from fluctuating dipoles and separated by a vacuum gap. Summing pairwise over all microscopic van der Waals interactions between the dipoles provides the macroscopic Casimir interaction between the slabs. On the other hand, we can determine the dielectric properties of the individual slabs as produced by the fluctuating dipoles. The specific boundary conditions due to the dielectric properties of the slabs influence the spectrum of the electromagnetic field confined in between. The change in the spectrum as a function of the separation of the slabs produces the Casimir force. This analogy between the Casimir and the van der Waals force is transparently brought out within a path integral formulation, see Fig. 2: Assume the system under consideration can be described by an action $\mathcal{S}[\mathbf{a},\mathbf{j}]$ depending on the gauge field $\mathbf{a}$ and a particle current $\mathbf{j}$. Carrying out the partial integration in the partition function $\mathcal{Z} = \int \mathcal{D}[\mathbf{a}]\mathcal{D}[\mathbf{j}]\, e^{-\mathcal{S}[\mathbf{j},\mathbf{a}]/\hbar}$ over the matter field $\mathbf{j}$ or over the gauge field $\mathbf{a}$, we obtain an effective action $\mathcal{S}_{\rm eff}$ describing the conjugate field alone: From the effective action $\mathcal{S}_{\rm eff}[\mathbf{a}]$ describing the gauge field we can derive the Casimir effect, while the interaction of the particle currents as described by $\mathcal{S}_{\rm eff}[\mathbf{j}]$ will give us the van der Waals attraction.

The Casimir- and van der Waals forces then can be related to one another via the pairwise summation of the interparticle forces[18]: Consider two d-dimensional homogeneous macroscopic dielectric bodies of density ρ with parallel interfaces separated by a distance R. The two bodies attract one another due to a microscopic particle-particle interaction $V_{\rm vdW} = \Lambda\, r^{-\alpha}$, $\alpha > d$, of the van der Waals type. The interaction energy can be written in the form

$$U(R) = L^{d-1}\rho^2 \int_R^\infty \mathrm{d}x \int_x^\infty \mathrm{d}x' \int_E \mathrm{d}^{d-1}y\, V_{\rm vdW}(r),$$

where E is a hypercube of size L^{d-1} parallel to the interface and $r = \sqrt{x^2+y^2}$, with $\mathbf{y}$ the $d-1$ dimensional in-plane coordinate, while x is the coordinate along the direction perpendicular to the plane. We then find for the Casimir force density[19] $f = -\partial_R U(R)/L^{d-1}$ the result

$$\begin{aligned} f &= \rho^2 \int_R^\infty \mathrm{d}x \int_E \mathrm{d}^{d-1}y\, V_{\rm vdW}(r) \\ &= \Lambda\rho^2\, \pi^{(d-1)/2} \frac{\Gamma[(1+\alpha-d)/2]}{\Gamma(\alpha/2)} \frac{1}{\alpha-d} \frac{1}{R^{\alpha-d}}. \end{aligned} \tag{1}$$

The result (1) then allows to infer the parameters Λ and α, characterizing the van der Waals interaction $V_{\rm vdW}$, from the macroscopic Casimir force density f.

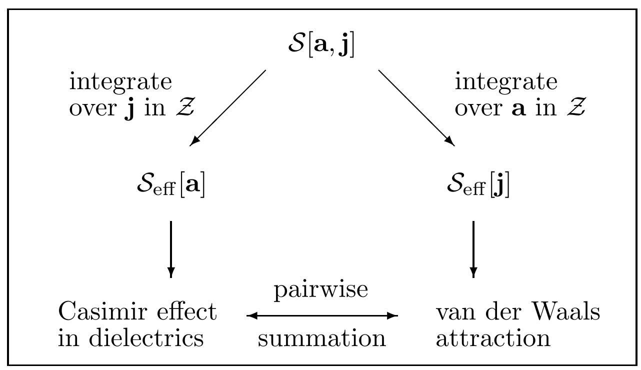

FIG. 2. Schematic relation between the Casimir effect and the van der Waals attraction.

After mapping the thermally fluctuating vortex matter to a system of 2D quantum charged bosons, we will arrive at an action $\mathcal{S}[\mathbf{j},\mathbf{A},\mathbf{a}]$ with two gauge fields $\mathbf{a}$ and $\mathbf{A}$, see Eq. (7). The Casimir effect we are interested in here is the one produced by the fake gauge field $\mathbf{a}$: The integration over the matter field $\mathbf{j}$ will produce the dielectric properties of the media, while the integration over the physical gauge field $\mathbf{A}$ renders the fake field massive and hence will always exponentially confine the Casimir force to finite distances.

III. BOSON REPRESENTATION OF VORTICES

We start from the London free energy in an isotropic superconductor[1]

$$\mathcal{F}[\mathbf{B}] = \frac{1}{8\pi} \int \mathrm{d}^3 r\, [\mathbf{B}^2 + \lambda^2(\nabla\times\mathbf{B})^2], \tag{2}$$

with $\mathbf{B} = \nabla \times \mathbf{A}$ the magnetic field and λ denoting the London penetration depth. In order to acquaint for the vortices, we add the current term $-(\mathbf{j} \cdot \mathbf{B})\Phi_0/8\pi$ with

$$\mathbf{j} = (\mathbf{J}, \rho) = \sum_\mu \mathbf{t}_\mu(z)\, \delta^{(2)}(\mathbf{R} - \mathbf{R}_\mu(z)), \qquad (3)$$

where $\mathbf{r} = (\mathbf{R}, z)$, $\mathbf{R} = (x, y) \in \mathbb{R}^2$, the coordinates $\mathbf{R}_\mu(z)$ denote the position, the vectors $\mathbf{t}_\mu = (\partial_z \mathbf{R}_\mu(z), 1)$ the direction of the vortex lines, and $\Phi_0 = hc/2e$ is the unit of flux. Ignoring screening, the interaction between the vortex lines is long ranged and thus can conveniently be expressed through a mediating gauge field $\mathbf{a}$. As usual, we introduce the gauge field $\mathbf{a}$ as an auxiliary field such that $\int \mathcal{D}\mathbf{a} \exp(-\beta \mathcal{F}'[\mathbf{a}]) = \exp(-\beta \mathcal{F})$, with $\beta = 1/T$ the inverse temperature[8] (we set the Boltzmann constant k_B to unity and fix the gauge through the condition $\nabla \cdot \mathbf{a} = 0$),

$$\mathcal{F}'[\mathbf{a}, \mathbf{A}, \mathbf{j}] = \int \mathrm{d}^3x \left[i\mathbf{a} \cdot \left(\mathbf{j} - \frac{1}{\Phi_0}(\nabla \times \mathbf{A}) \right) + \frac{1}{8\pi}(\nabla \times \mathbf{A})^2 + \frac{1}{2g^2}(\nabla \times \mathbf{a})^2 \right], \qquad (4)$$

where $g^2 = 4\pi\varepsilon_0$ and the line energy $\varepsilon_0 = (\Phi_0/4\pi\lambda)^2$ is the basic energy scale in the problem. In (4), we have accounted for screening by introducing back into the model the real gauge field $\mathbf{A}$.

The anisotropy of uniaxial layered materials is most conveniently introduced through a rescaling of the scalar fake magnetic field $b = (\nabla \times \mathbf{a})_z$[20,21],

$$(\nabla \times \mathbf{a})_z \to \frac{1}{\varepsilon}(\nabla \times \mathbf{a})_z,$$

where the anisotropy factor $\varepsilon = \sqrt{m_{ab}/m_c}$ is determined through the effective masses perpendicular (m_c) and parallel (m_{ab}) to the ab-plane (the fake electric field $\mathbf{e}_\perp = (e_y, -e_x) = -(\nabla \times \mathbf{a})_{xy}$ remains unchanged; the subscript 'xy' identifies the planar component of a vector, $\mathbf{a} = (\mathbf{a}_{xy}, a_z)$).

We map the statistical mechanics of the vortex system to the imaginary time quantum mechanics of 2D bosons through the replacements[5,7,8] $z \to \tau$ (imaginary boson time), $\beta^{-1} \to \hbar^B$ (the boson's Planck constant), $\mathcal{F}' \to \mathcal{S}$ (the boson action), and $1/\varepsilon \to c$ (the light velocity in the boson system). The boson partition function is given by

$$\mathcal{Z} = \int \mathcal{D}[\{\mathbf{R}_\mu\}]\mathcal{D}[\mathbf{A}]\mathcal{D}[\mathbf{a}] e^{-\mathcal{S}[\{\mathbf{R}_\mu\}, \mathbf{A}, \mathbf{a}]/\hbar^B}, \qquad (5)$$

with $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\mathrm{int}}$ and

$$\mathcal{S}_0[\{\mathbf{R}_\mu\}] = \int \mathrm{d}\tau \sum_\mu \left[\frac{m}{2} (\partial_\tau \mathbf{R}_\mu(\tau))^2 - \mu^B \right], \qquad (6)$$

$$\mathcal{S}_{\mathrm{int}}[\{\mathbf{R}_\mu\}, \mathbf{A}, \mathbf{a}] = \int \mathrm{d}\tau \mathrm{d}^2R \left[i\mathbf{a} \cdot \left(\mathbf{j} - \frac{1}{\Phi_0}(\nabla \times \mathbf{A}) \right) + \frac{1}{8\pi}(\nabla \times \mathbf{A})^2 + \frac{1}{2g^2} \left((\nabla \times \mathbf{a})_{xy}^2 + \frac{1}{\varepsilon^2}(\nabla \times \mathbf{a})_\tau^2 \right) \right]. \qquad (7)$$

Here, $\mu^B = H\Phi_0/4\pi - \varepsilon_0 \ln(\lambda/\xi)$ is the chemical potential of the bosons, $\rho = B/\Phi_0$ is the boson density, and $m = \varepsilon_0$ is the boson mass (we have introduced a term $-\mathbf{BH}/4\pi$ to account for the external magnetic field $\mathbf{H}$ producing the vortices in the superconductor; ξ denotes the planar coherence length). In the thermodynamic limit, we have $L_z \to \infty$, corresponding to $T^B = \hbar^B/L_z \to 0$, i.e., we are interested in the ground state physics of the boson system. Note that the free boson action $\mathcal{S}_0$ contains the bare mass m due to the vortex core energy. The retarded self-interaction of the bosons via their gauge fields then produces the mass renormalization $m \to m^B = \varepsilon_l(k_z)$, where $\varepsilon_l(k_z)$ is the dispersive line tension[8] of the vortex lines.

IV. VAN DER WAALS INTERACTION

To set the stage, we briefly review the derivation of the van der Waals interaction as presented in Ref. 2; this will allow us to fix some flaws in the previous derivation and will provide us with a check on the results for the Casimir force derived later.

For a simple qualitative analysis we consider two vortices in a layered superconductor (with layers separated by the distance d). Ignoring the coupling between the layers, the fluctuating pancake vortices interact via a logarithmic potential $V(R) \approx 2\epsilon_0 \ln(R/\lambda)$. Second order perturbation theory then provides us with a van der Waals interaction $-V_{\mathrm{vdW}} \sim (T/d)(\lambda/R)^4$, the energy scale T/d being set by the driving action of the thermal fluctuations. At long distances $R > d/\varepsilon$ a finite interlayer coupling changes this result as the Josephson interaction reduces the cutoff $1/d$ on the k_z modes to the new value $1/\varepsilon R$; the van der Waals interaction crosses over to $V_{\mathrm{vdW}} \sim -(T/\varepsilon\lambda)(\lambda/R)^{-5}$. The two results have their analogue in the van der Waals attraction between neutral atoms, where the interaction potential exhibits a crossover from a r^{-6} at short to a r^{-7} behavior at large distances[22].

We proceed with the derivation of the long range van der Waals interaction between vortex lines/bosons. Following the scheme in Fig. 2, we integrate over the gauge fields $\mathbf{A}$ and $\mathbf{a}$ in the partition function (5) and obtain the effective current-current interaction

$$\mathcal{F}[\mathbf{j}] = \frac{\varepsilon_0}{2} \int \mathrm{d}^3r\, \mathrm{d}^3r'\, j_\alpha(\mathbf{r}) V^{\mathrm{int}}_{\alpha\beta}(\mathbf{r} - \mathbf{r}') j_\beta(\mathbf{r}) \qquad (8)$$

(we return to the more natural statistical mechanics notation in this section, $\mathcal{S}_{\mathrm{eff}} \to \mathcal{F}$). Inserting the expression (3) for the currents, we can cast Eq. (8) into the standard form[8]

$$\mathcal{F}[\{\mathbf{r}_\mu\}] = \frac{\varepsilon_0}{2} \sum_{\mu,\nu} \int \mathrm{d}r_{\mu\alpha} \mathrm{d}r'_{\nu\beta} V^{\mathrm{int}}_{\alpha\beta}(\mathbf{r}_\nu - \mathbf{r}'_\mu), \qquad (9)$$

with the vortex positions $\mathbf{r}_\mu = (\mathbf{R}_\mu, z)$ and the interaction potential $V^{\rm int}_{\alpha\beta}$, conveniently expressed within a Fourier representation,

$$V^{\rm int}_{\alpha\beta}(\mathbf{r}-\mathbf{r}') = 4\pi\lambda^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} V^{\rm int}_{\alpha\beta}(\mathbf{k}), \qquad (10)$$

with

$$V^{\rm int}_{\alpha\beta}(\mathbf{k}) = \frac{e^{-(\xi^2K^2+\xi_c^2k_z^2)}}{1+\lambda^2k^2} \times \left[\delta_{\alpha\beta} - \frac{(\lambda_c^2-\lambda^2)K_{\perp\,\alpha}K_{\perp\,\beta}}{1+\lambda^2k^2+(\lambda_c^2-\lambda^2)K^2}\right]. \qquad (11)$$

Here $\mathbf{k} = (\mathbf{K}, k_z) = (k_x, k_y, k_z)$, $\mathbf{K}_\perp = (k_y, -k_x)$ and $\lambda_c = \lambda/\varepsilon$, $\xi_c = \varepsilon\xi$. The free energy (9) can be split into the self energy part $\mathcal{F}^{\,0}$ ($\mu = \nu$) and the interaction part $\mathcal{F}^{\,\rm int}$ ($\mu \neq \nu$). Restricting ourselves to two vortices a distance R apart, we obtain

$$\mathcal{F}^{\,\rm int} = \frac{\Phi_0^2}{4\pi}\int \frac{\mathrm{d}^3k}{(2\pi)^3} dz_1 dz_2\, e^{i\mathbf{K}[\mathbf{R}+\mathbf{u}_1-\mathbf{u}_2]} e^{ik_z(z_1-z_2)} \times [V^{\rm int}_{zz}(\mathbf{k}) + t_{1\alpha}(z_1)t_{2\beta}(z_2)V^{\rm int}_{\alpha\beta}(\mathbf{k})], \qquad (12)$$

where we have split the vortex positions $\{\mathbf{R}_1, \mathbf{R}_2\}$ into a mean field part $\{\mathbf{R}_1^0, \mathbf{R}_2^0\} = \{\mathbf{0}, \mathbf{R}\}$ and a fluctuating part $\{\mathbf{u}_1, \mathbf{u}_2\}$. Up to a constant, the free energy is given by

$$F(R) = -T\ln Z(R) = -T\ln\langle \exp[-\beta\mathcal{F}^{\rm int}]\rangle_0. \qquad (13)$$

The average $\langle\ldots\rangle_0$ has to be taken with respect to the self-energy $\mathcal{F}^{\,0}$ of the free vortices. Performing a cumulant expansion, we obtain the effective vortex-vortex interaction in the form $L_z V_{\rm eff} \approx \langle\mathcal{F}^{\,\rm int}\rangle_0 - [\langle(\mathcal{F}^{\,\rm int})^2\rangle_0 - \langle\mathcal{F}^{\,\rm int}\rangle_0^2]/2T$, where L_z denotes the sample thickness. Splitting into longitudinal (to the induction, $\mathcal{F}_\parallel$) and transverse ($\mathcal{F}_\perp$) parts, the longitudinal interaction produces the standard repulsive vortex-vortex interaction[1], to lowest order in $\mathbf{u}_\mu$,

$$V_{\rm rep}(R) = 2\varepsilon_0 K_0(R/\lambda), \qquad (14)$$

while higher orders in $\mathbf{u}_\mu$ merely renormalize the prefactor. The transverse part produces the van der Waals interaction[2]

$$V_{\rm vdW} = -\frac{\langle\mathcal{F}_\perp\mathcal{F}_\perp\rangle_0}{2TL_z}. \qquad (15)$$

Using the decomposition

$$\langle t_{1\alpha}(-k_z)t_{2\beta}(k_z)t_{1\alpha'}(-k_z')t_{2\beta'}(k_z')\rangle_0 = 2\pi L_z\delta(k_z-k_z')\delta_{\alpha\alpha'}\delta_{\beta\beta'}\frac{\langle t^2(k_z)\rangle_0^2}{4},$$

we can reduce the average in (15) to the simpler form

$$\langle\mathcal{F}_\perp\mathcal{F}_\perp\rangle_0 = \frac{\Phi_0^2L_z}{64\pi^2}\int\frac{\mathrm{d}k_z}{2\pi}[V^{\rm int}_{\alpha\beta}(\mathbf{R},k_z)]^2\langle t^2(k_z)\rangle_0^2, \qquad (16)$$

with the partial Fourier transform

$$V^{\rm int}_{\alpha\beta}(\mathbf{R},k_z) = \int\frac{\mathrm{d}^2K}{(2\pi)^2}V^{\rm int}_{\alpha\beta}(\mathbf{K},k_z)e^{i\mathbf{KR}}. \qquad (17)$$

In strongly anisotropic and layered material the single vortex mean squared amplitude of fluctuations $\langle t^2(k_z)\rangle_0^2$ is limited by the electromagnetic interaction through the dispersive elasticity $\varepsilon_l(k_z)$,

$$\langle t^2(k_z)\rangle_0 = \frac{2T}{\varepsilon_l(k_z)}, \qquad (18)$$

with

$$\varepsilon_l(k_z) = \frac{\varepsilon_0}{2\lambda^2k_z^2}\ln(1+\lambda^2k_z^2). \qquad (19)$$

The evaluation of the partial Fourier transform (17) is carried out in Appendix A and making use of the results (A1) and (A2) in the limit $\varepsilon = 0$, we find the van der Waals interaction in the decoupled limit

$$V^{\rm dc}_{\rm vdW} = -\frac{4\varepsilon_0}{\ln^2(\pi\lambda/d)}\frac{T}{d\varepsilon_0}\left(\frac{\lambda}{R}\right)^4. \qquad (20)$$

The interlayer distance d provides the large k_z cutoff on the thermal fluctuations of the individual lines.

In the continuous anisotropic case the single vortex elasticity is still dominated by its electromagnetic contribution. At intermediate distances $\lambda < R < d/\varepsilon$ we recover again the result (20) as the Josephson coupling is not effective yet. At large distances $\lambda, d/\varepsilon < R < \lambda_c$, however, the interlayer coupling becomes important through cutting off the k_z modes at $1/\varepsilon R$ and we find the result

$$V^{\rm ret}_{\rm vdW} = -\frac{(171\pi/256)\varepsilon_0}{\ln^2(\pi\lambda_c/R)}\frac{T}{\varepsilon\lambda\varepsilon_0}\left(\frac{\lambda}{R}\right)^5. \qquad (21)$$

These results coincide with the ones found by Blatter and Geshkenbein[2] up to the numerical prefactor as they missed a factor 2 in Eq. (18), as well as a term in the partial Fourier transform which contributes to the result (21) to order $O(\varepsilon^{-1})$. The same mistakes show up in their results for the attraction of the vortex to the sample surface, Eqs. (19) and (20) of Ref. 2. We give the correct expressions here:

$$V^{\rm s,dc}_{\rm vdW} = -\frac{\varepsilon_0/2}{\ln(\pi\lambda/d)}\frac{T}{d\varepsilon_0}\left(\frac{\lambda}{R}\right)^2 \qquad (22)$$

for the decoupled case, and

$$V^{\rm s}_{\rm vdW} = -\frac{(3/16)\varepsilon_0}{\ln(\pi\lambda/\varepsilon R)}\frac{T}{\varepsilon\lambda\varepsilon_0}\left(\frac{\lambda}{R}\right)^3 \qquad (23)$$

in the continuous anisotropic situation.

V. CASIMIR FORCE

In this section, we derive the Casimir force between two parallel cubes of vortex matter and then show, via the method of pairwise summation, that the results coincide with those found by means of the van der Waals approach.

Again, we begin with a simple analysis based on dimensional estimates, instructing us what to expect. Consider two d-dimensional hypercubes of size L^d separated by a distance $R \ll L$, see Fig. 1. We wish to determine the Casimir force f per unit area acting between the interfaces. Being due to the zero point fluctuations of the gauge field, the only relevant dimensional quantities entering the expression for the force are Planck's constant $\hbar^B$, the velocity of light c, and the distance R between the slabs. The combination $\hbar^B c/R^{d+1}$ then provides us with a dimensionally correct expression for the force; in two dimensions we then find $f \sim \hbar^B c/R^3$ for the retarded Casimir effect. In the non-retarded case, $c \to \infty$ and we have to replace the combination c/R by the frequency scale ω where the matter becomes transparent (for the vortex matter in a layered superconductor this 'frequency' is given by the layer separation d, $\omega \sim 1/d$). Dimensional estimates then give us for the force per unit length in two dimensions $f \sim \hbar^B\omega/R^2$. In order to derive the corresponding microscopic inter-particle potential from these estimates, we apply the method of pairwise summation and obtain the result $V_{\rm vdW} \propto 1/\varepsilon R^5$ in the retarded case and $V_{\rm vdW} \propto 1/R^4$ in the non-retarded limit. These simple estimates produce the correct power laws for the van der Waals potential; in order to find the correct sign and the complete prefactors, we have to go through the detailed analysis below.

A. Formalism

The Casimir energy is given through the difference in the sum over all cavity modes minus the free field contribution[17,23]. While the calculation is rather straightforward for the simple case of a metallic cavity, an appropriate formalism has to be set up to treat more general configurations involving dielectric media.

We start from a quadratic Lagrangian density $\mathcal{L} = \mathbf{a}\mathbf{G}^{-1}\mathbf{a}$ in d dimensions, with $\mathbf{a}(\mathbf{x},\tau)$ a vector boson field with r components and $\mathbf{G}$ the Green function matrix. The partition function $\mathcal{Z}$ is expressed through the usual imaginary time path integral formalism

$$\mathcal{Z} = \int \mathcal{D}[\mathbf{a}]\, \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^d x \int_0^{\hbar/T} d\tau\, \mathcal{L}[\mathbf{a}]\right] = [\det \mathbf{G}]^{-1/2}, \tag{24}$$

and the free energy $F = -T\ln\mathcal{Z}$ is given by

$$F = \frac{T}{2}\mathrm{Tr}\ln\mathbf{G}^{-1}. \tag{25}$$

We consider the classic parallel-plate geometry with isotropic media separated by a gap of width R along the x_d-direction. For each polarization $\nu \in \{1,\dots,r\}$ we determine the eigenstates obeying the boundary conditions at the $(d-1)$-dimensional hypersurfaces placed at 0 and R. The individual modes are characterized by their polarization ν, the (Matsubara) frequency $\xi_s = 2\pi sT/\hbar$, $s \in \mathbb{Z}$, the transverse wave vector $\mathbf{q} \in \mathbb{R}^{d-1}$, and the longitudinal wave vector k_n, $n \in \mathbb{N}$. Given the set $(\nu, \xi_s, \mathbf{q})$, the discrete wave vectors k_n are obtained as the solutions of the boundary condition $D_{\{\alpha\}}(k) = 0$, where $\{\alpha\}$ is a short hand for the other indices $(\nu, \xi_s, \mathbf{q})$ (the function $D_{\{\alpha\}}(k)$ derives from the determinant associated with the set of boundary conditions; for the classic Casimir effect, $D(k) = \sin(kR)$ with R the distance between the metal plates). Expressing the trace through all the indices we have

$$F = \frac{TL^{d-1}}{2}\sum_{\nu=1}^{r}{\sum_{s=0}^{\infty}}'\int\frac{\mathrm{d}^{d-1}q}{(2\pi)^{d-1}}\sum_{D_{\{\alpha\}}(k)=0}\ln G^{-1}_{\{\alpha\}}(k), \tag{26}$$

where the prime on the sum indicates that we count the $s = 0$ term with a weight 1/2. In (26) we have made explicit the boundary condition at the hypersurfaces separating the media from the gap.

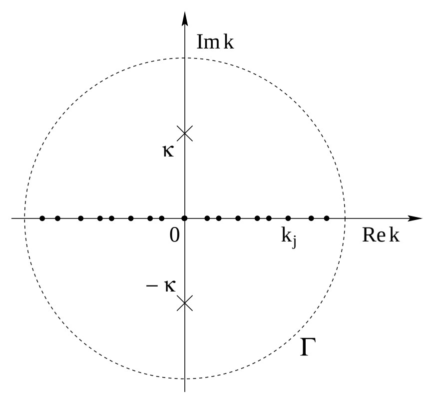

FIG. 3. Contour and distribution of poles in the complex k-plane as they appear in the expression (27). The poles produced by the boundary condition $D(k) = 0$ are marked by the solid dots on the real axis. The zeros of $g(k)$ are denoted by the crosses. All zeroes are distributed symmetrically with respect to the origin, a consequence of the symmetry $G(k) = G(-k)$.

We make use of Cauchy's theorem to perform the sum over the longitudinal momenta k_n: We first rewrite the sum in the form

$$\sum_{D_{\{\alpha\}}(k)=0}\ln[G^{-1}_{\{\alpha\}} + y] = \int \mathrm{d}y \sum_{D_{\{\alpha\}}(k)=0}\frac{1}{G^{-1}_{\{\alpha\}} + y}.$$

Next, we let the boundary condition $D_{\{\alpha\}}(k) = 0$ produce the desired sum in a Cauchy loop integral in the

complex k plane, see Fig. 3. With $g(k) = (G^{-1}_{\{\alpha\}}(k) + y)$ we have

$$\frac{1}{2\pi i}\oint_\Gamma \frac{\partial_k D(k)}{D(k)}\frac{1}{g(k)} \tag{27}$$
$$= \sum_{D(k)=0}\frac{1}{g(k)} + \sum_{g(k)=0}\frac{\partial_k D(k)}{D(k)}\frac{1}{\partial_k g(k)},$$

where it is understood that both D and g further depend on the other indices $\{\alpha\}$ as well as on the parameter y in the case of g. We define the function $w(\{\alpha\};y)$ as the zeroes of the expression $g_{\{\alpha\}}(k) = G^{-1}_{\{\alpha\}}[w(\{\alpha\};y)]+y = 0$. Taking the derivative of the last equation with respect to y, $\partial_y g = \partial_y G^{-1} + 1 = \partial_k G^{-1}\partial_y w + 1$, and using the result in (27), we obtain the desired result

$$\sum_{D_{\{\alpha\}}(k)=0}\frac{1}{G^{-1}_{\{\alpha\}}+y} = 2\frac{\partial}{\partial y}\ln D_{\{\alpha\}}[w(\{\alpha\};y)], \tag{28}$$

where we have used that the Green function exhibits the proper asymptotics $G(k) \sim O(k^2)$ and the symmetry $G(k) = G(-k)$. Inserting the sum on k, Eq. (28) with $y = 0$, back into the free energy expression (26) we obtain

$$\mathcal{F} = T\sum_{\{\alpha\}}\ln D_{\{\alpha\}}[w(\{\alpha\})], \tag{29}$$

where

$$\sum_{\{\alpha\}}[\ldots] \equiv L^{d-1}\sum_{\nu=1}^{r}\sum_{s=0}^{\infty}{}' \int \frac{\mathrm{d}^{d-1}q}{(2\pi)^{d-1}}[\ldots]$$

denotes the sum/integral over all modes $\{\alpha\} = (\nu, \xi_s, \mathbf{q})$.

The above formal manipulations require a regularization such that $D[\xi_s \to \infty] \to 1$. We carry out the appropriate renormalization by the subtraction of the free energy of the free field without boundary conditions, corresponding to replacing the partition function through the ratio $\mathcal{Z} \to \mathcal{Z}/\mathcal{Z}_0$, with $\mathcal{Z}_0$ the partition function of the free field. For the latter, we define the 'boundary condition' D_0 in such a way as to produce the fraction R/L of the free field energy in the volume L^d via the expression (29); for the classic Casimir problem, $D_0(k \pm i\delta) \propto \lim_{L\to\infty}\{\sin[(k\pm i\delta)L]\}^{R/L}$. With $D^{\rm ren} = D/D_0$ we finally obtain the regularized free energy of the Casimir problem[17]

$$\mathcal{F}^{\rm ren} = T\sum_{\{\alpha\}}\ln D^{\rm ren}_{\{\alpha\}}[w(\{\alpha\})]. \tag{30}$$

We illustrate the use of this formalism with a brief derivation of the classic Casimir result, the attraction between two parallel metallic plates of size $L \times L$ and a distance R apart. The dispersion relation for the free electromagnetic field is $G^{-1}(\xi_s, \mathbf{q}, k) = \xi_s^2/c^2 + q^2 + k^2$, such that the function $w|_{y=0}$ takes the form $w(\xi_s, \mathbf{q}) = i\sqrt{\xi_s^2/c^2 + q^2}$. The boundary condition requiring the fields to vanish at 0 and R can be cast into the form $D(k) = \sin(kR) = 0$, producing the modes with a longitudinal k vector $k_n = 2\pi n/R$. The 'boundary condition' describing the free field takes the form $D_0(k\pm i\delta) = (\mp 1/2i)\exp(\mp ikR)$, such that $D^{\rm ren}(k\pm i\delta) = 1 - \exp(\mp 2ikR)$, where $\delta \to 0^+$. Combining the results for $D^{\rm ren}$ and w, we have to carry out the mode summation over the logarithm of

$$D^{\rm ren}[w(\xi_s, \mathbf{q})] = \left[1 - \exp\left(-2R\sqrt{\frac{\xi_s^2}{c^2} + q^2}\right)\right]^2, \tag{31}$$

where taking the square accounts for the two polarization modes of the electromagnetic field (the single polarization mode at $k = 0$ has been properly taken into account here). The force density f between the plates is given through the derivative of the energy (30) with respect to R,

$$f = -\frac{1}{L^2}\frac{\partial \mathcal{F}^{\rm ren}}{\partial R} = -\frac{\hbar}{\pi R}\int_0^\infty \mathrm{d}q q\int_0^\infty \frac{\mathrm{d}\xi}{2\pi}$$
$$\times \frac{2R\sqrt{\xi^2/c^2 + q^2}}{\exp(2R\sqrt{\xi^2/c^2+q^2}) - 1}, \tag{32}$$

where we have replaced the Matsubara sum through an integral over complex frequencies ξ. The remaining integrals are easily evaluated by changing to polar coordinates $q = \rho\cos\phi$, $\xi/c = \rho\sin\phi$ and we obtain the classic result due to Casimir[9]

$$f = -\frac{\pi^2}{240}\frac{\hbar c}{R^4}. \tag{33}$$

B. Application to 2D bosons

In the following, we apply the above formalism to the 2D charged bosons described by the action $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\rm int}$ as given in Eqs. (6) and (7). Following the scheme outlined in Fig. 2, we first have to integrate over the currents $\mathbf{J} \sim \partial_\tau \mathbf{R}$. We split the boson positions $\mathbf{R}_\mu(\tau)$ into the mean field part $\mathbf{R}^0_\mu(\tau)$ and a fluctuating part $\mathbf{u}_\mu(\tau)$, $\mathbf{R}_\mu(\tau) = \mathbf{R}^0_\mu(\tau) + \mathbf{u}_\mu(\tau)$. In carrying out the integration over the boson positions $\mathbf{R}_\mu(\tau)$ we can ignore the fluctuating part $\mathbf{u}_\mu(\tau)$ in the gauge field $\mathbf{a}$ as the latter is smooth on the scale of the amplitude $\mathbf{u}_\mu(\tau)$,

$$\int \mathcal{D}[\mathbf{u}_\mu]\exp\bigg\{-\frac{1}{\hbar^B}\int \mathrm{d}\tau\sum_\mu\Big[\frac{m}{2}(\partial_\tau \mathbf{u}_\mu)^2$$
$$+i(\partial_\tau \mathbf{u}_\mu, 1)\cdot \mathbf{a}(\mathbf{R}^0_\mu, \tau)\Big]\bigg\}$$
$$\propto \exp\left[-\frac{1}{\hbar^B}\int \mathrm{d}^2R\mathrm{d}\tau\left(\frac{1}{2}\Pi(\mathbf{R})a^2_{xy}(\mathbf{R}) + i\rho(\mathbf{R})a_\tau(\mathbf{R})\right)\right],$$

where

$$\Pi(\mathbf{R}) = \frac{\rho(\mathbf{R})}{m} \quad \text{and} \quad \rho(\mathbf{R}) = \sum_\mu \delta^{(2)}(\mathbf{R} - \mathbf{R}^0_\mu)$$

denote the polarizability and the density of the 2D bosons, respectively. Second, we have to integrate over the true gauge field $\mathbf{A}$ (we have fixed the gauge to $\nabla \cdot \mathbf{a} = 0$). The resulting term $(1/2g^2\lambda^2)a^2$ in the action then renders the fake gauge field $\mathbf{a}$ massive (physically, this mass term expresses the finite range λ of the interaction between the charged bosons/vortices). Third, we introduce the free Green function matrix $\mathbf{G}_0$ for the gauge field, in Fourier representation,

$$\mathbf{G}_0^{-1} = \begin{bmatrix} \mathbf{g}_{xy}^{-1} & 0 \\ 0 & g_\tau^{-1} \end{bmatrix},$$

$$\text{with} \quad (\mathbf{g}_{xy})^{-1}_{\alpha\beta} = k^2\delta_{\alpha\beta} + (c^2-1)K^2 P_{\alpha\beta}(\mathbf{K}),$$
$$g_\tau^{-1} = k^2,$$

and the transverse projector $P_{\alpha\beta} = \delta_{\alpha\beta} - K_\alpha K_\beta/K^2$. Combining the results of the above three steps, we arrive at the desired effective action for the fake gauge field $\mathbf{a}$,

$$\mathcal{S}_{\text{eff}}[\mathbf{a}] = \int \mathrm{d}^2R\mathrm{d}\tau \left[\mathbf{a}_{xy}\frac{1}{2g^2}\left(\mathbf{g}_{xy}^{-1} + \frac{1}{\lambda^2} + g^2\Pi\right)\mathbf{a}_{xy} + a_\tau \frac{1}{2g^2}\left(g_\tau^{-1} + \frac{1}{\lambda^2}\right)a_\tau + i\rho a_\tau\right]. \tag{34}$$

The Euler-Lagrange equations of this action determine the (imaginary time) field equations for the transverse and longitudinal parts of $\mathbf{a}$ (we remind that $\lambda_c = c\lambda = \lambda/\varepsilon$),

$$\left(\frac{1}{c^2}\partial_\tau^2 + \nabla_{xy}{}^2 - \frac{g^2\rho}{m^{\scriptscriptstyle B}c^2} - \frac{1}{\lambda_c^2}\right)\mathbf{a}_{xy} = \left(1 - \frac{1}{c^2}\right)\nabla_{xy}(\nabla_{xy}\cdot\mathbf{a}_{xy}) = 0 \tag{35}$$

and

$$\left(\nabla_{xy}^2 - \frac{1}{\lambda^2}\right)a_\tau = i\rho g^2, \tag{36}$$

with $\nabla_{xy} = (\partial_x, \partial_y)$. In the last equation of (35) we have made use of the gauge condition $\nabla\cdot\mathbf{a} = \nabla_{xy}\cdot\mathbf{a}_{xy} + \partial_\tau a_\tau = 0$ and have ignored the time dependence in the longitudinal field component a_τ, see below.

The longitudinal field a_τ is generated by the source $g^2\rho/4\pi = \varepsilon_0\rho$ and has no dynamics on its own. Placing two bosons at a distance R, a simple integration gives the vortex-vortex interaction (c.f., Eq. (14))

$$V_{\text{rep}}(R) = 2\varepsilon_0 K_0(R/\lambda). \tag{37}$$

Here, we ignore the fluctuations of the vortex position in the calculation of a_τ as they merely produce a small renormalization of the prefactor in the repulsive potential. Similarly, we can neglect such corrections in the calculation of the transverse modes below and set the r. h. s. of (35) equal to zero.

The dynamical transverse field $\mathbf{a}_{xy}$ generates the Casimir force. Here, we have in mind a geometry as sketched in Fig. 1, with two parallel cubes of vortex matter of density ρ, separated by a vortex free region of thickness R. We choose the x-axis to lie perpendicular to the 'plates', the z-axis is directed along the vortices. Within the boson language we deal with two parallel planes of 2D bosons lying in the xy-plane, the direction along the vortices now transforming to the imaginary time coordinate (below, we consider the limit $T^{\scriptscriptstyle B} \to 0$, implying vortex lines of infinite length, $L_z \to \infty$). Note that in the present 2D case we have to consider only one polarization mode for the transverse gauge field $\mathbf{a}_{xy}$, a consequence of the gauge condition $\nabla\cdot\mathbf{a} = 0$.

Going over to Fourier space we can cast (35) into the form (we remind the reader that $c = 1/\varepsilon$; q and k are the wave vectors along and perpendicular to the 'plates')

$$\left[\frac{\xi^2}{c^2}\left(1 + \frac{g^2\rho}{m^{\scriptscriptstyle B}\xi^2}\right) + q^2 + k^2 + \frac{1}{\lambda_c^2}\right]\mathbf{a}_{xy} = 0, \tag{38}$$

from which we obtain the Green function $G_0^{-1}(\xi, q, k) = \epsilon_{\scriptscriptstyle V}(\xi)\xi^2/c^2 + q^2 + k^2 + \lambda_c^{-2}$ and the function w,

$$w(\xi, q) = i\sqrt{\frac{\epsilon_{\scriptscriptstyle V}(\xi)}{c^2}\xi^2 + q^2 + \frac{1}{\lambda_c^2}}. \tag{39}$$

Here, we have defined the 'dielectric' constant

$$\epsilon_{\scriptscriptstyle V}(\xi) = 1 + \frac{g^2\rho}{m^{\scriptscriptstyle B}\xi^2} = 1 + \frac{8\pi\lambda^2\rho}{\ln(1+\lambda^2\xi^2)}, \tag{40}$$

where we go over to vortex parameters in the last equation ($m^{\scriptscriptstyle B} \to \varepsilon_l$) and make use of the expression (19) for the dispersive vortex elasticity ε_l.

Next, we have to formulate the boundary conditions in terms of the zeroes of the function $D_{\{\alpha\}}(k) = 0$. The translational invariance in the y-direction allows for a plane-wave Ansatz $\mathbf{a}_{xy}(y) \propto e^{iqy}$, while the sequence of 'dielectric' and 'vacuum' regions along the x-axis, requires to match the plane waves $\mathbf{a}_{xy}(x) \propto e^{ik(x)x}$, with (see (38))

$$k_l = k(x<0) = \sqrt{k^2 - \frac{(\epsilon_{\scriptscriptstyle V}(\xi)-1)}{c^2}\xi^2},$$
$$k_0 = k(0<x<R) = k,$$
$$k_r = k(R<x) = \sqrt{k^2 - \frac{(\epsilon_{\scriptscriptstyle V}(\xi)-1)}{c^2}\xi^2}. \tag{41}$$

The gauge condition $\nabla_{xy}\cdot\mathbf{a}_{xy}$ is satisfied through the Ansatz

$$\mathbf{a}_{xy} = \alpha(x)\begin{pmatrix} q \\ -k(x) \end{pmatrix} e^{i(k(x)x+qy)} + \beta(x)\begin{pmatrix} q \\ k(x) \end{pmatrix} e^{-i(k(x)x+qy)},$$

with piecewise constant amplitudes $\alpha(x)$ and $\beta(x)$. The six coefficients are determined by the boundary conditions at $x = 0$ and $x = R$, requiring the parallel electric field $\mathbf{e}_\parallel$ and the magnetic field b to be continuous, see Appendix B. Two further conditions force the fields to vanish at $x = \pm L$, $L \to \infty$. Requiring the determinant of the resulting 6×6 matrix problem to vanish, we find

$$\begin{aligned} D[\xi, q, k] &= e^{-ik_l L} e^{-ik_0 R} e^{-ik_r L} \\ &\Big[1 - \frac{(1 + (\lambda\xi)^{-2})k_l - (\epsilon_V + (\lambda\xi)^{-2})k}{(1 + (\lambda\xi)^{-2})k_l + (\epsilon_V + (\lambda\xi)^{-2})k} \\ &\times \frac{(1 + (\lambda\xi)^{-2})k_r - (\epsilon_V + (\lambda\xi)^{-2})k}{(1 + (\lambda\xi)^{-2})k_r + (\epsilon_V + (\lambda\xi)^{-2})k} e^{2ikR}\Big]. \end{aligned} \tag{42}$$

For the free field, we find

$$D_0(k + i\delta) = e^{-ik_l L} e^{-ik_0 R} e^{-ik_r L}, \tag{43}$$

and combining the above results for w, D, and D_0, Eqs. (39), (42), and (43), we can construct the function $D^{\text{ren}}[w(\xi, q)]$ and obtain the formal expression for the free energy (30) of the vortex Casimir problem. We note that the cut introduced through the definitions of k_l and k_r in Eq. (41) disappears from the boundary condition D and hence does not contribute to the loop integral (27). As a result, the free energy expression (30) for the Casimir free energy remains valid.

C. Casimir Interaction

In the present two dimensional situation the expression for the Casimir energy Eq. (30) per length reads

$$\frac{\mathcal{F}^{\text{ren}}(R)}{L} = \hbar^B \int_0^\infty \frac{\mathrm{d}q}{2\pi} \int_0^\infty \frac{\mathrm{d}\xi}{\pi} \ln D^{\text{ren}}[w(\xi, q)], \tag{44}$$

and produces the Casimir force per length L

$$\begin{aligned} f &= -\frac{1}{L}\frac{\partial \mathcal{F}^{\text{ren}}(R)}{\partial R} \\ &= -\frac{\hbar^B}{i\pi^2} \int_0^\infty \mathrm{d}q \int_0^\infty \mathrm{d}\xi \, w(\xi, q) \\ &\times \Big[\frac{(1 + (\lambda\xi)^{-2})w_l + (\epsilon_V + (\lambda\xi)^{-2})w}{(1 + (\lambda\xi)^{-2})w_l - (\epsilon_V + (\lambda\xi)^{-2})w} \\ &\times \frac{(1 + (\lambda\xi)^{-2})w_r + (\epsilon_V + (\lambda\xi)^{-2})w}{(1 + (\lambda\xi)^{-2})w_r - (\epsilon_V + (\lambda\xi)^{-2})w} e^{-2iwR} - 1\Big]^{-1}. \end{aligned} \tag{45}$$

We introduce the new variables

$$\begin{aligned} p &= \sqrt{1 + c^2 q^2/\xi^2}, \\ s &= \sqrt{p^2 + (\lambda\xi)^{-2}}, \\ s_\epsilon &= \sqrt{(\epsilon_V - 1) + p^2 + (\lambda\xi)^{-2}}, \end{aligned}$$

and arrive at the simplified expression

$$\begin{aligned} f = &-\frac{\hbar^B}{\pi^2} \int_1^\infty \mathrm{d}p \int_0^\infty \mathrm{d}\xi \frac{sp\xi^2}{c^2\sqrt{p^2 - 1}} \\ &\left[\left(\frac{(1 + (\lambda\xi)^{-2})s_\epsilon + (\epsilon_V + (\lambda\xi)^{-2})s}{(1 + (\lambda\xi)^{-2})s_\epsilon - (\epsilon_V + (\lambda\xi)^{-2})s}\right)^2 e^{2\xi sR/c} - 1\right]^{-1}. \end{aligned} \tag{46}$$

Eq. (46) gives the two dimensional version for massive photons of the Casimir force between dielectric media, first derived by Lifshitz[24] for the conventional 3D situation.

In the following we will be interested in the situation where the 'dielectric' media are dilute (dilute vortex matter at small magnetic induction $B \ll \Phi_0/\lambda^2$). In this case, we expand (46) in the small correction $\epsilon_V - 1$ giving the deviation of the dielectric constant from its vacuum value. Furthermore, if the integral is dominated by large frequencies ξ we can ignore the mass term everywhere. Under these conditions, we approximate $s \approx p$ and $s_\varepsilon \approx (\epsilon_V - 1)(1/2p - p) + \epsilon_V p$, such that $s_\varepsilon - \epsilon_V p \approx (\epsilon_V - 1)(1/2p - p)$, while $s_\varepsilon + \epsilon_V p \approx 2p$, and obtain the simplified force expression (the weak logarithmic disperion in ϵ_V is approximated through an appropriate constant)

$$f = -\frac{\hbar^B(\epsilon_V - 1)^2}{16\pi^2}\int_1^\infty \mathrm{d}p \int_0^\infty \mathrm{d}\xi \frac{\xi^2(2p^2 - 1)^2}{c^2p^2\sqrt{p^2 - 1}} e^{2\xi pR/c} \tag{47}$$

In (46), (47) we have to account for three relevant frequency- or length scales. The first is introduced by the discrete nature of our superconductor: the layered structure limits the ξ-integration through the frequency cutoff $\xi_d = \pi/d$ (this corresponds to the medium becoming transparent at high frequencies $\xi > \xi_d$). The second frequency scale is introduced through the dispersion in the dielectric constant: at low frequencies $\xi < \xi_\lambda = 1/\lambda$, the dielectric constant takes the form $\epsilon_V(\xi \to 0) \to 8\pi\rho/\xi^2$ and the dielectric response changes over to a mass renormalization. The third frequency scale $\xi_R = 1/R\varepsilon$ appears through the exponential $\exp(2\xi sR/c) > \exp(2\xi R\varepsilon)$ in the integrand of (46); as ξ goes beyond ξ_R the exponent cuts off the integrand in (46).

Thus, depending on the frequency ξ, the media are either transparent, give a dielectric response, or produce a mass renormalization. And depending on the position of ξ_R with respect to these response regimes, we will find a different behavior of the Casimir force. Below, we will analyze the various regimes and use the pairwise summation technique to show that the results are in agreement with those obtained for the van der Waals interaction potential in Sec. IV above.

1. *Intermediate Distances $\lambda < R < d/\varepsilon$*

At intermediate distances $\lambda < R < d/\varepsilon$, we have $\xi_d < \xi_R$ such that the cutoff on the ξ integral is given by ξ_d. As the integral is dominated by large values of ξ

we can ignore the mass terms $1/\lambda^2\xi^2$ everywhere. The exponential $\exp(2\xi sR/c)$ restricts the p-integral to values $p < d/\varepsilon R$, admitting large values of p such that we can drop the corrections to p^2 in (47), e.g., $p^2 - 1 \approx p^2$. Transforming p to the new variable $\gamma = 2R\xi\, p/c$, we find the Casimir force on intermediate scales

$$\begin{aligned} f &= -\frac{\hbar^B(\epsilon_V - 1)^2}{16\pi^2 R^2}\int_0^{\xi_d} \mathrm{d}\xi \int_0^\infty \mathrm{d}\gamma\gamma e^{-\gamma} \\ &= -\frac{(\epsilon_V - 1)^2}{16\pi^2}\frac{\hbar^B\xi_d}{R^2}. \end{aligned} \tag{48}$$

Using the result (1) of the pairwise summation and inserting the expression (40) for the dielectric constant, we obtain the van der Waals interaction describing the decoupled limit,

$$V_{\mathrm{vdW}}^{\mathrm{dc}} = -\frac{4\varepsilon_0}{\ln^2(\pi\lambda/d)}\frac{T}{d\varepsilon_0}\left(\frac{\lambda}{R}\right)^4, \tag{49}$$

in agreement with the result (20) (note that the density ρ cancels in the final result for the van der Waals interaction).

2. Large Distances $\lambda, d/\varepsilon < R < \lambda/\varepsilon$

At large distances, the ξ-integral is cutoff on the scale ξ_R, i.e., the Casimir force is limited by the distance R through the exponential $\exp(2\xi sR/c)$ rather than by the transparency of the material. We still can ignore the mass terms in (46) as we assume that $\xi_\lambda < \xi_R$, providing us with a large regime for the ξ-integration where $\xi > 1/\lambda$. Transforming the energy variable ξ to $\gamma = (2R\xi p)/c$, we obtain the following expression for the Casimir force in the low density limit,

$$\begin{aligned} f &= -\frac{\hbar^B c(\epsilon_V - 1)^2}{8\pi^2 R^3}\int_0^\infty \mathrm{d}\gamma\gamma^2 e^{-\gamma}\int_1^\infty \mathrm{d}p\frac{(2p^2-1)^2}{16p^5\sqrt{p^2-1}} \\ &= -\frac{19(\epsilon_V - 1)^2}{1024\pi}\frac{\hbar^B c}{R^3}. \end{aligned} \tag{50}$$

Following the usual scheme, this result produces the retarded van der Waals interaction

$$V_{\mathrm{vdW}}^{\mathrm{ret}} = -\frac{(171\pi/256)\varepsilon_0}{\ln^2(\pi\lambda/\varepsilon R)}\frac{T}{\lambda\varepsilon\varepsilon_0}\left(\frac{\lambda}{R}\right)^5, \tag{51}$$

in agreement with Eq. (21).

3. Very Large Distances $\lambda/\varepsilon < R$

The situation at very large distances $R > \lambda/\varepsilon = \lambda_c$ is most conveniently analyzed in the original formulation (45). The mode summation is limited by the exponential $\exp(2iwR)$, implying the following restrictions on the integration variables ξ and q: $\xi/c < 1/\sqrt{R\lambda_c} < 1/\lambda_c$ and $q < 1/\sqrt{R\lambda_c} < 1/\lambda_c$. The integral $\int_0^\infty \mathrm{d}q \int_0^\infty \mathrm{d}\xi\,[\ldots]$ then is given by the area $c/R\lambda_c$ times the $q, \xi \to 0$ limit of the integrand. In the limit $\xi \to 0$ the mass terms are relevant and the dispersion in the dielectric constant (40) produces the additional mass renormalization

$$\frac{1}{\lambda_c} \to \sqrt{\frac{1}{\lambda_c^2} + \frac{8\pi}{c^2}\rho} \equiv M. \tag{52}$$

The Casimir force then decays exponentially following the law

$$f = -\frac{1}{2\pi}\frac{\hbar^B c}{\lambda_c^2 R}\frac{(1-\lambda_c M)^2}{(1+\lambda_c M)^2}\, e^{-2R/\lambda_c}. \tag{53}$$

For small densities, we can expand $\lambda_c M \approx 1 + 4\pi\lambda^2\rho$ and obtain for the screened van der Waals interaction in the regime $\lambda/\varepsilon < R$ the expression (note that the pairwise summation formula (1) has to be modified for the exponential factor in the interaction VvdW)

$$V_{\mathrm{vdW}}^{\mathrm{sc}} = -4\sqrt{\pi}\varepsilon_0\frac{T\varepsilon^4}{\varepsilon_0\lambda}\left(\frac{\lambda_c}{R}\right)^{3/2}\exp(-2R/\lambda_c). \tag{54}$$

VI. DISCUSSION

We have derived the Casimir force between two bodies of vortex matter and have inferred from the result the strength of the van der Waals interaction between vortex lines via the method of pairwise summation. While the calculation of the Casimir force is carried out in the boson formulation, the van der Waals attraction is determined in the vortex picture. The agreement between the results is once more an illustration of the equivalence of the two formalisms[5,25].

The physical implications of these results lead to interesting modifications of the B-T phase diagram of layered/anisotropic type II superconductors[2]: The attraction between the lines produces a generic vortex solid of density $\rho \sim 10^{-2}/\lambda^2$ at low temperatures. An interesting problem appearing in this context is the accurate determination of the entropic repulsion between the vortex lines, the latter giving an important contribution to the Gibbs free energy. Various approaches to this problem have been discussed by Blatter and Geshkenbein[21], by Volmer, Mukherji, and Nattermann[4], and by Volmer and Schwartz[27]. The transition from the Meissner state to the van der Waals vortex solid takes place via a sharp first-order transition at the lower critical field H_{c_1}. The concurrent phase separation then can be observed in a decoration experiment, though very clean samples are required. Furthermore, the phase separation leads to the concentration of the vortex lines into dense regimes separated by vortex free regions. The interaction between the vortex domains then is given by the Casimir force

calculated here, thus providing a natural realization of the physics described in this paper.

A further remark is in place concerning the physical character of the van der Waals interaction discussed here. The usual van der Waals interaction between neutral atoms arises from dipolar fluctuations in the charge distribution of the atoms, resulting in a force which is mediated through the scalar (longitudinal) potential, at least within the non-retarded regime at small distances. Comparing to the van der Waals force between the 2D charged bosons/vortex lines discussed here, we note that the elementary objects do not have an internal structure producing a fluctuating dipole. The van der Waals interaction then arises from *current* fluctuations and thus involves the (transverse) vector field. In principle, such a 'transverse' van der Waals force is also present in an electron gas: while the longitudinal repulsive interaction is screened on the Thomas-Fermi length, the transverse attractive interaction of the van der Waals type due to fluctuating currents survives at long distances (and in principle induces a superconducting instability). However, the transverse van der Waals force in an electronic system involves the small parameter v_F/c (v_F is the Fermi velocity), rendering the effect small. In the 2D boson system discussed here, the light velocity is given by the anisotropy of the superconductor, $c = 1/\varepsilon$, and thus is a tunable parameter.

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APPENDIX A: PARTIAL FOURIER TRANSFORMS

In this section we sketch, by way of example, the calculation of $V_{xx}^{\rm int}(\mathbf{R}, k_z)$. We start from Eq. (11) and set $\alpha = \beta = x$ (here, we ignore the exponential cutoff function due to the vortex core; the cases $\alpha = \beta = y$ and $\alpha = x$, $\beta = y$ follow trivially)

$$V_{xx}^{\rm int}(\mathbf{K}, k_z) = \frac{1}{1+\lambda^2 k^2}\left(1 - \frac{(\lambda_c^2 - \lambda^2)k_y^2}{1+\lambda_c^2 K^2 + \lambda^2 k_z^2}\right).$$

Performing a contour-integral over k_x, we obtain

$$\begin{aligned} V_{xx}^{\rm int}(\mathbf{R}, k_z) &:= I_1 + I_2 \\ &= \frac{1}{4\pi\lambda a^2}\int \mathrm{d}k_y Y(\lambda, k_y) e^{-(R_x/\lambda)Y(\lambda,k_y)} e^{ik_y R_y} \\ &\quad - \frac{\lambda_c}{4\pi a^2}\int \mathrm{d}k_y \frac{k_y^2}{Y(\lambda_c, k_y)} e^{-(R_x/\lambda_c)Y(\lambda_c,k_y)} e^{ik_y R_y}, \end{aligned}$$

with $Y(v,w) = \sqrt{a^2 + v^2 w^2}$ and $a^2 = 1 + \lambda^2 k_z^2$. Carrying out the integral[28] over k_y in I_1 we find

$$I_1 = \frac{1}{2\pi a^2}\frac{\partial^2}{\partial R_x^2} K_0\left(\frac{aR}{\lambda}\right).$$

The second integral I_2 can be calculated in the same manner,

$$I_2 = \frac{1}{2\pi a^2}\frac{\partial^2}{\partial R_y^2} K_0\left(\frac{aR}{\lambda_c}\right).$$

Here, K_0 denotes the 0-th order modified Bessel functions of the second kind and $R^2 = R_x^2 + R_y^2$. Collecting results and carrying out the derivatives, we obtain the desired result. Here, we quote the final expressions for the case $\mathbf{R} = (R, 0)$:

$$V_{xx}^{\rm int}(R, k_z) = \frac{1}{2\pi a^2}\left\{\frac{a^2}{\lambda^2}\left[K_0\left(\frac{aR}{\lambda}\right) + \frac{\lambda}{aR}K_1\left(\frac{aR}{\lambda}\right)\right] - \frac{a}{\lambda_c R}K_1\left(\frac{aR}{\lambda_c}\right)\right\}, \quad \text{(A1)}$$

$$V_{yy}^{\rm int}(R, k_z) = \frac{1}{2\pi a^2}\left\{\frac{a^2}{\lambda_c^2}\left[K_0\left(\frac{aR}{\lambda_c}\right) + \frac{\lambda_c}{aR}K_1\left(\frac{aR}{\lambda_c}\right)\right] - \frac{a}{\lambda R}K_1\left(\frac{aR}{\lambda}\right)\right\}. \quad \text{(A2)}$$

The third component $V_{xy}^{\rm int}(R, k_z)$ vanishes for $R_y = 0$. The important terms in (A1) and (A2) are those involving the large screening length λ_c in the argument of the Bessel functions. The first term $\propto K_0$ in (A2) has been missed in Ref. 2.

APPENDIX B: 2D ELECTRODYNAMICS

We sketch the main features of the 2D real time massive electrodynamics underlying the present work (we denote the mass by $1/\lambda$ and restrict ourselves to the isotropic case). We introduce the metric $\mathbf{g} = \mathrm{diag}(-1,-1,+1)$ and define the gauge field $\mathbf{a} = (\mathbf{a}_{xy}, a_t)$. In addition, we define the derivatives $\partial^\alpha = (-\nabla_{xy}, \partial_t)$, where $\nabla_{xy} = (\partial_x, \partial_y)$. The field tensor $f^{\alpha\beta}$ is given by

$$f^{\alpha\beta} = \partial^\alpha a^\beta - \partial^\beta a^\alpha = \begin{pmatrix} 0 & -b & e_x \\ b & 0 & e_y \\ -e_x & -e_y & 0 \end{pmatrix}, \quad \text{(B1)}$$

and its dual takes the form $f^{*\alpha} = \epsilon^{\alpha\beta\gamma} f_{\beta\gamma} = (e_y, -e_x, -b)$ ($\epsilon^{\alpha\beta\gamma}$ is the antisymmetric tensor). The Lagrangian of the massive field coupled to an external current j^α is given by

$$\mathcal{L} = -\frac{1}{4g^2} f_{\alpha\beta} f^{\alpha\beta} + \frac{1}{2\lambda^2 g^2} a_\alpha a^\alpha + a_\alpha j^\alpha, \quad \text{(B2)}$$

with g^2 the coupling constant. This Lagrangian produces an imaginary time action

$$\mathcal{S}[\mathbf{a}] = \int \mathrm{d}\tau \mathrm{d}^2 R\left[\frac{1}{2g^2}(\nabla\times\mathbf{a})^2 + \frac{1}{2g^2\lambda^2}\mathbf{a}^2 + \mathbf{aj}\right]. \quad \text{(B3)}$$

The transformation from real time to imaginary time is obtained via the formal rules[29] $t \to -i\tau$, $\partial_t \to i\partial_\tau$, $a_t \to$

$-ia_\tau$. The magnetic field remains unchanged, while the electric field transforms to $\mathbf{e} \to i\left[(\nabla\times\mathbf{a})_{xy}\right]_\perp = i\mathbf{e}$. The free field real time Lagrangian $\left(\mathbf{e}^2 - b^2\right)/2g^2$ then goes over into the simple form $\left(\mathbf{e}^2 + b^2\right)/2g^2 = (\nabla\times\mathbf{a})^2/2g^2$ within the imaginary time formalism.

The functional derivative of Eq. (B2) provides us with the inhomogeneous Maxwell equations (the homogeneous one, $\partial_\alpha f^{*\alpha} = -\partial_t b + \mathrm{rot}\,\mathbf{e} = 0$, follows from the antisymmetric structure of $f^{\alpha\beta}$),

$$\partial_\alpha f^{\alpha\beta} + \frac{1}{\lambda^2} a^\beta = g^2 j^\beta. \tag{B4}$$

Making use of Gauss' and Stokes' theorems for a loop encircling the boundary between two media, we obtain the boundary conditions $\mathbf{e}_\parallel =$ continuous and $b =$ continuous.

The quantization of the theory can be developed via the Gupta-Bleuler formalism within the Lorentz gauge $\partial_\alpha a^\alpha = 0$. In the case of a massive theory this method leads to two polarization modes, while in a massless theory the gauge invariance reduces the number of polarization modes by one. Within the present theory, the coupling of $\mathbf{a}$ to the real $\mathbf{A}$ field in the form $i\mathbf{a}\cdot(\nabla\times\mathbf{A})/\Phi_0$ produces a gauge invariant mass term $\mathbf{a}\cdot(\mathbf{a} - \mathbf{k}\cdot(\mathbf{k}\cdot\mathbf{a})/k^2)/2\lambda^2 g^2$. As the gauge invariance is conserved in our theory we end up with a single polarization mode.

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